Wavelet Analysis for Time and Frequency Transfer

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Abstract—In this manuscript, we discuss the application of wavelet transform to analysis the phase or frequency noise of the time and frequency transfer (TFT) signals. We find that the wavelet transform can extract certain time-varying characteristics of the TFT residual noise, which may be used for noise compensation with the help of machine learning to break through the limitation of the traditional method for phase compensation.

Index Terms—wavelet transform, time and frequency transfer (TFT), wavelet variance, noise compensation, machine learning

I. Introduction

The usual method to evaluate the stability of the time and frequency transfer (TFT) is to calculate Allan variance of the residual noise [1]. However, the result sometimes is divergent if certain types of noise is included in the residual noise. Various different v ersions o f Allan variance are designed to make it capable of disposing more complicated cases [2], [3]. In the meantime, we notice that, Allan variance is a special case of the wavelet variance. And Allan variance can scarcely provide any useful message for phase compensation. The traditional method of phase compensation used for TFT is the two-way or round-trip method [4], [5]. But these methods are based on the assumption that the transfer link is symmetric for the signals transmitting in both directions of the link. In practice, it is impossible to ensure the symmetry of the transfer link, and a lot of factors can make the link asymmetric. The asymmetry can serious affects the stability of the transfer link with the traditional method for phase compensation [6], [7]. Here we show that except as a method to evaluate the stability of the system by calculating the wavelet variance, the wavelet transformation can also extract certain time-varying characteristics of the TFT residual noise. These characteristics are supposed to be connected to the asymmetry caused by certain apparatus. Hence, if each apparatus is characterized by certain pattern of noise, we actually can compensate the noise according to these noise pattern with the help of, e.g., the machine learning. This could be helpful to break through the limitation of the traditional method for phase compensation.

II. Methods and results

A wavelet is a small wave that grows and decays essentially in a limited time period [8]. Any function that integrates to zero and is square integrable can be used to define a wavelet. It is a function with time and scale parameters [9], i.e.,

$$\psi_{\lambda,t}(u) = \frac{1}{\sqrt{\lambda}}\psi\left(\frac{u-t}{\lambda}\right), \tag{1}$$

where, t and λ are called center time and scale, respectively. The continuous wavelet transformation (CWT) of a time series $x(t)$ is defined as

$$W(\lambda, t) = \int_{-\infty}^{\infty} \psi_{\lambda,t}(u)x(u)\mathrm{d}u. \tag{2}$$

One also has the inverse CWT

$$x(t) = \frac{1}{C_\psi}\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^2}\int_{-\infty}^{\infty} \mathrm{d}u W(\lambda, u)\psi_{\lambda,u}(t). \tag{3}$$

if the admissibility condition is satisfied, i.e.,

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\tilde{\psi}(f)|^2}{f}\mathrm{d}f < \infty, \tag{4}$$

where, $\tilde{\psi}(f)$ is the Fourier transformation of $\psi(t)$.

In practice, orthogonal wavelet is usually used for data analysis. It is a wavelet with discrete scale and center time, i.e.,

$$\psi_{m,n}(t) = 2^{-m/2}\psi(2^{-m}t - n), \quad m, n = 0,\ 1,\ 2, \cdots, \tag{5}$$

and satisfied the orthogonal condition

$$\langle\psi_{m,n}(t), \psi_{k,l}(t)\rangle = \int_{-\infty}^{\infty} \psi_{m,n}(t)\psi_{k,l}\mathrm{d}t = \delta_{mk}\delta_{nl}. \tag{6}$$

The discrete wavelet can be generally constructed by the multiresolution analysis (MRA) theory [9], where, two orthogonal subspaces are defined, i.e., the wavelet subspace $W_m = \{\psi_{m,n},\ n = 0,\ \pm 1,\ \pm 2, \cdots\}$, and the scaling subspace $V_m = \{\phi_{m,n},\ n = 0,\ \pm 1,\ \pm 2, \cdots\}$. They have the following property

$$V_m \oplus W_m = V_{m-1}. \tag{7}$$

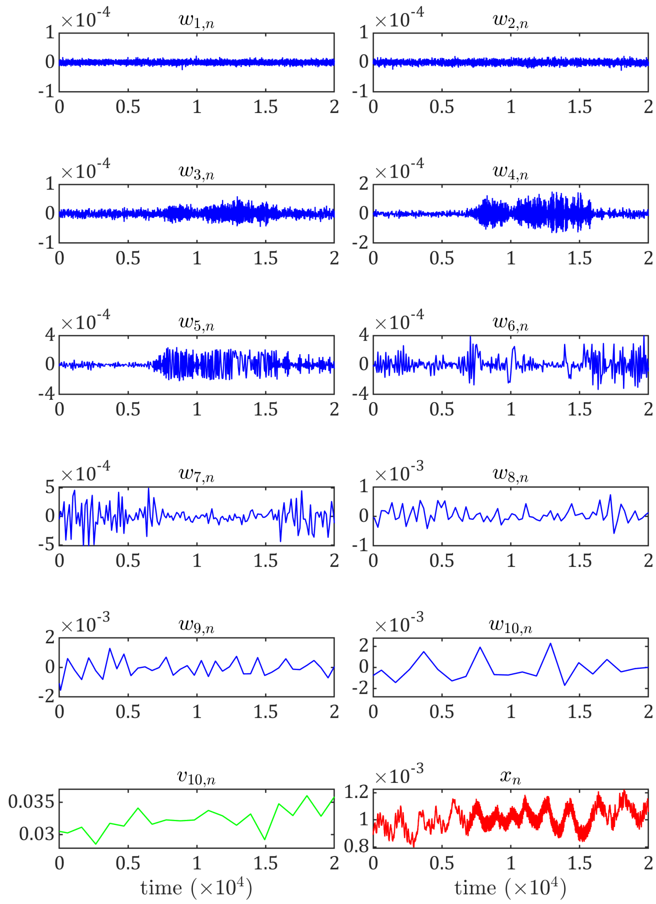

Fig. 1. D(4) wavelet transform

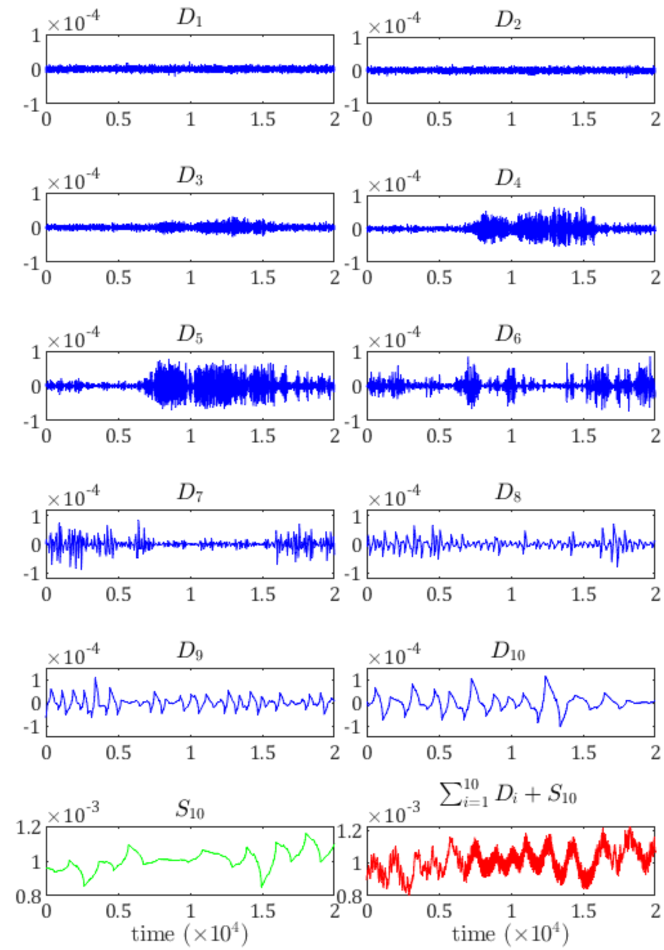

Fig. 2. Inverse D(4) wavelet transform

Hence, one can express $\phi_{m,n}$ and $\psi_{m,n}$ in terms of $\psi_{m-1,n}$. Suppose

$$\phi_{m,n} = \sum_k g_k \phi_{m-1,2n+1-k}, \tag{8}$$

$$\psi_{m,n} = \sum_k h_k \phi_{m-1,2n+1-k}, \tag{9}$$

$\{g_k,\ k = 0,\ 1,\ 2, \cdots,\ L-1\}$ and $\{h_k,\ k = 0,\ 1,\ 2, \cdots,\ L-1\}$ are called scaling filter and wavelet filter, respectively. They have the following relation

$$h_l = (-1)^l g_{L-1-l}. \tag{10}$$

Hence, the wavelet functions can be constructed by properly choosing the filter, e.g., the filter of Haar wavelet is described by 2 nonzero coefficients, and the Daubechies or D(4) wavelet is described by 4 nonzero coefficients. For a continuous time series $x(t)$, the discrete wavelet transformation (DFT) is defined as

$$v_{m,n} = \langle x, \phi_{m,n} \rangle = \sum_k g_k v_{m-1,2n+1-k}, \tag{11}$$

$$w_{m,n} = \langle x, \psi_{m,n} \rangle = \sum_k h_k v_{m-1,2n+1-k}. \tag{12}$$

It is usually impossible to get a simple and explicit expression of the wavelet or scaling function. Therefore, we need to use the pyramid algorithm displayed by euqations (11) and (12) to complete the calculation. And in practice, the time series $x(t)$ is sampled, and we only have the discrete time series x_n. We can assume that the sample process is represented by

$$x_n = \langle x, \phi_{0,n} \rangle = v_{0,n}, \tag{13}$$

or simply define the 1^{st} stage DWT by

$$v_{1,n} = \sum_k g_k x_{m-1,2n+1-k}, \tag{14}$$

$$w_{1,n} = \sum_k h_k x_{m-1,2n+1-k}. \tag{15}$$

As the discrete time series x_n has a limited length (denoted by N), the definition should be further modified as

$$v_{1,n} = \sum_k g_k x_{m-1,2n+1-k \mod N}, \tag{16}$$

$$w_{1,n} = \sum_k h_k x_{m-1,2n+1-k \mod N}, \tag{17}$$

where, N should be an even number, and note that the dimension of $v_{1,n}$ is $N/2$. Similarly, if we continue the

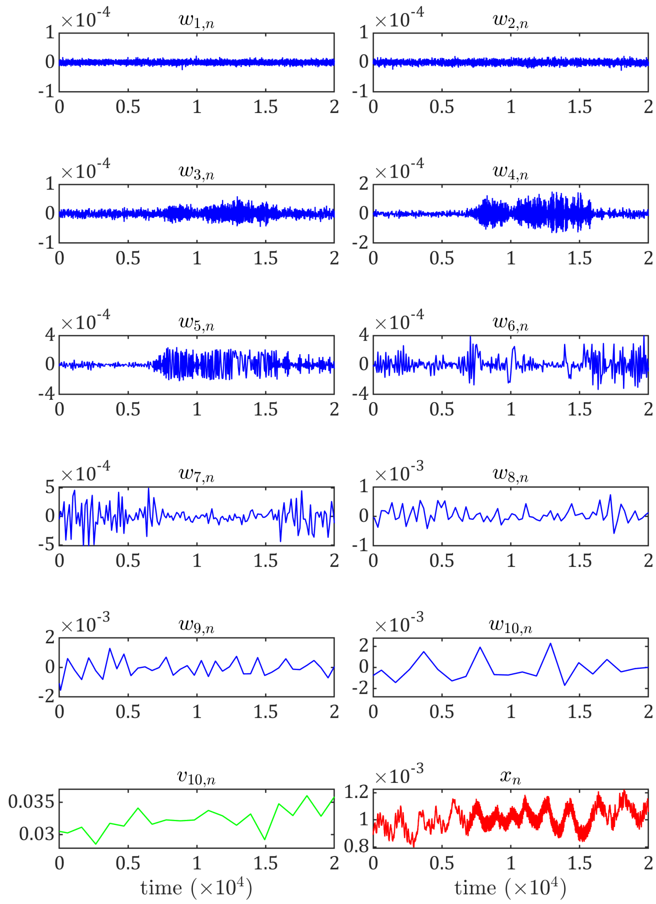

Fig. 3. LA(16) wavelet transform

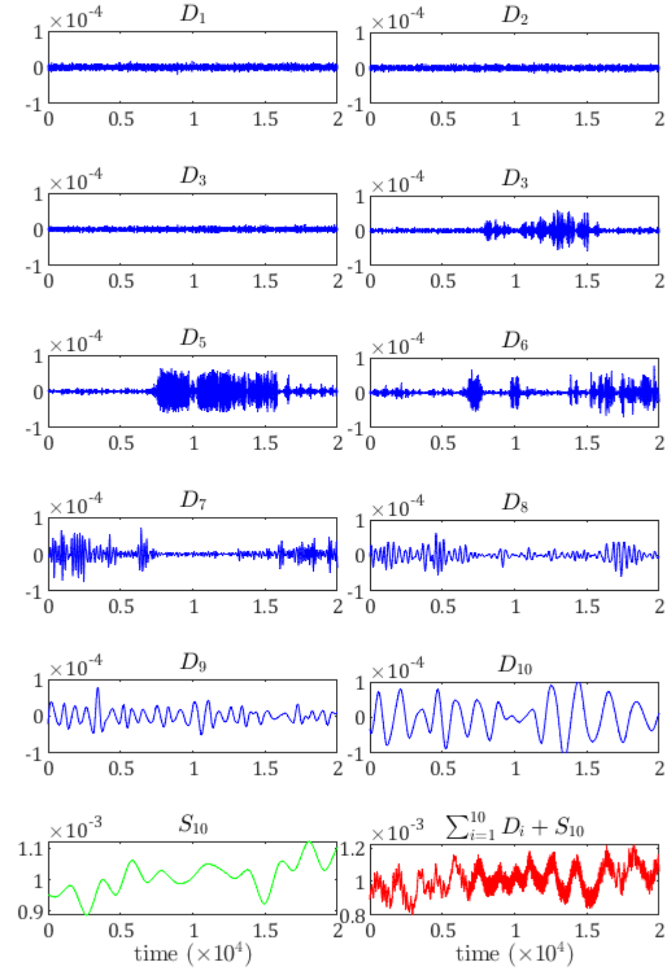

Fig. 4. Inverse LA(16) wavelet transform

DWT to the m-th stage, N should be the multiple of 2^m, and the transformation should be

$$v_{1,n} = \sum_k g_k x_{m-1, 2n+1-k \mod N/2^{m-1}}, \tag{18}$$

$$w_{1,n} = \sum_k h_k x_{m-1, 2n+1-k \mod N/2^{m-1}}, \tag{19}$$

To calculate the inverse DWT, we can first define the upsampling of a discrete series $v_{m,n}$ (or $w_{m,n}$) by

$$\begin{cases} v^{\uparrow}_{1,2n} = 0, \\ v^{\uparrow}_{1,2n+1} = v^{\uparrow}_{1,n+1}, \end{cases} \qquad n = 0,\ 1, \cdots,\ \frac{N}{2^m}. \tag{20}$$

Define the following two inverse transformation

$$\mathcal{S}^{-1}(v_{m,n}) = \sum_k h_k v^{\uparrow}_{m,n+k \mod N/2^{m-1}}, \tag{21}$$

$$\mathcal{W}^{-1}(v_{m,n}) = \sum_k g_k v^{\uparrow}_{m,n+k \mod N/2^{m-1}}, \tag{22}$$

then the inverse DWT can be expressed as

$$v_{m-1,n} = \mathcal{S}^{-1}(v_{m,n}) + \mathcal{W}^{-1}(w_{m,n}). \tag{23}$$

Continue the calculation, one can derive

$$x_n = \sum_{k=1}^{m} \mathcal{S}^{-k+1}\mathcal{W}^{-1}(w_{k,n}) + \mathcal{S}^{-m}(v_{m,n}) = \sum_{k=1}^{m} D_k + S_m, \tag{24}$$

where

$$D_k = \mathcal{S}^{-k+1}\mathcal{W}^{-1}(w_{k,n}), \tag{25}$$

$$S_m = \mathcal{S}^{-m}(v_{m,n}). \tag{26}$$

We calculate the DWT of an optical fiber frequency transfer phase noise with Daubechies wavelet D(4) and least asymmetric wavelet LA(16) up to the 10-th stage, and their inverse transformation, shown in Fig.1-Fig.4, respectively. The first 11 subfigures of each figure are the wavelet/scaling coefficients or their inverse transformation coefficients at different scales. The final one is the discrete time series of the phase noise or the sum of all the inverse transformation results. The coefficients of the scaling filters of the D(4) wavelet and LA(16) wavelet are list in [8].

III. Discussion

It is shown by $w_{4,n}$ and $w_{5,n}$ (or D_4 and D_5) that an apparent interruption happens around the time of

$5000 \sim 15000$ with a characteristic time scale of $2^4 \sim 2^5$. From $w_{7,n}$ and $w_{8,n}$ (or D_7 and D_8), we can see that a noise with a characteristic time scale of $2^7 \sim 2^8$ appears in the rest of time. It also can be observed from D_{10} an obvious noise with characteristic time scale 2^{10} exists in the whole time. When we sum $D_i,\ i = 1, \cdots,\ 10$ and S_{10}, it is exactly the original signal. Hence, they can be taken as a multiresolution analysis (MRA) of the original signal at different time scale. When comparing the transformation results of the two wavelets, we can find that LA(16) wavelet may offer a better "resolution" than D(4). On the one hand, the noise around the time of $5000 \sim 15000$ simultaneously appears in $w_{3,n}$, $w_{4,n}$ and $w_{5,n}$ of the D(4) wavelet, while it only appears in $w_{4,n}$ and $w_{5,n}$ of the LA(16) wavelet. On the other hand, LA(16) wavelet has more smooth inverse transformation than D(4) wavelet. It is perhaps due to the fact that LA(16) filter (16 nonzero coefficients) is longer than D(4)'s (4 nonzero coefficients). We should note that, if we continue the transformation in higher stage, MRA will offer more details about the phase noise.

IV. Conclusion

We demonstrate that the wavelet transformation can offer a good multiresolution analysis (MRA) for the frequency transfer phase noise. It is helpful for us to trace the origin of the noise, and thus, to modify the system. In our further work, we attempt to apply the noise features displayed by MRA for phase compensation.

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